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ASSESSMENT OF THE SLIDING FAILURE PROBABILITY OF A CONCRETE GRAVITY DAM

2nd INTERNATIONAL WEEK ON RISK ANALYSIS AS APPLIED TO DAM SAFETY AND DAM SECURITY

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CONTENTS

1.- INTRODUCTION.

The objective is to show different methods of analysis to estimate the conditional probability of failure by sliding along the dam-foundation contact on a concrete gravity dam. These methods can be classified as Level 1, Level 2 and Level 3 methods.

If our project variables ($X_1,X_2,...,X_n$) are considered as random variables, it is possible to define the strength function $r(x_1,x_2,...,x_n)$ and the load function $s(x_1, x_2,...,x_n)$ and write the limit state equation as follows:

$$g^*(x_1, x_2,..., x_n) = g(x_1, x_2,..., x_n) - 1 = \frac{r(x_1, x_2,..., x_n)}{s(x_1, x_2,..., x_n)} - 1 = 0 \qquad \text{(Eq. 1)}$$

According to this, the failure domain in the n-dimensional space is defined as all the possible values ($x_1,x_2,...,x_n$) that verify the condition:

$$g^*(x_1, x_2,..., x_n) \leq 0 \qquad \text{(Eq. 2)}$$

and the safety domain is defined as all possible values ($x_1,x_2,...,x_n$) that verify the condition:

$$g^*(x_1, x_2,..., x_n) > 0 \qquad \text{(Eq. 3)}$$

According to the concept of the probability density function, the probability of a single n-dimensional point ($x_1,x_2,...,x_n$) to be in the failure domain defined by $g^*(x_1,x_2,...,x_n)$, is calculated as the integral over the failure domain of the joint probability density function of all random variables:

$$P_f\left[g^*(x_1, x_2,..., x_n) \leq 0\right] = \int_{g^*(x_1, x_2,..., x_n) \leq 0} f_{X_1,X_2,...,X_n}(x_1, x_2,..., x_n) dx_1 dx_2 ... dx_n \qquad \text{(Eq. 4)}$$

As long as the joint probability density function and the integration domain are defined with precision, and the integral can be calculated, Equation (4) provides a value for probability that is mathematically exact.

The methods for failure probability estimation can be grouped in different levels (Mínguez [1]):

Level 1: Method of safety factors. Does not provide probability of failure. Uncertainty is measured by arbitrary factors.

Level 2: Second Moment Methods. The probability of failure can be obtained under some assumptions. Only the first two moments (mean and standard deviation) of the joint probability density function $f_{X1,X2,...,Xn}(x_1,x_2,...,x_n)$ are used. Eventually, also the failure domain $g^*(x_1,x_2,...,x_n)$ is approximated.

Level 3: Exact Methods. These methods provide the probability of failure, as they work with all the information of the joint probability density function. Integration is carried out by means of specific methods.

Table 1. Levels of reliability analysis.

LEVEL	Method of calculation	Probability distributions	Limit state equations	Treatment of uncertainty	Output
Level 1	Code calibration with level 2 or level 3 methods	Not used	Linear	Arbitrary factors	Coefficients
Level 2	Second order algebra	Normal distributions only	Linear or aprox. Linear	Can be included as normal distribution	Probability of failure
Level 3	Transformations	Equivalent normal distributions	Linear or aprox. Linear	Can be included	Probability of failure
	Numerical integration and simulation	Any distribution	Any form	Random variables	

2.- CASE STUDY.

Dam geometry is depicted in Figure 1 and in Table 2.

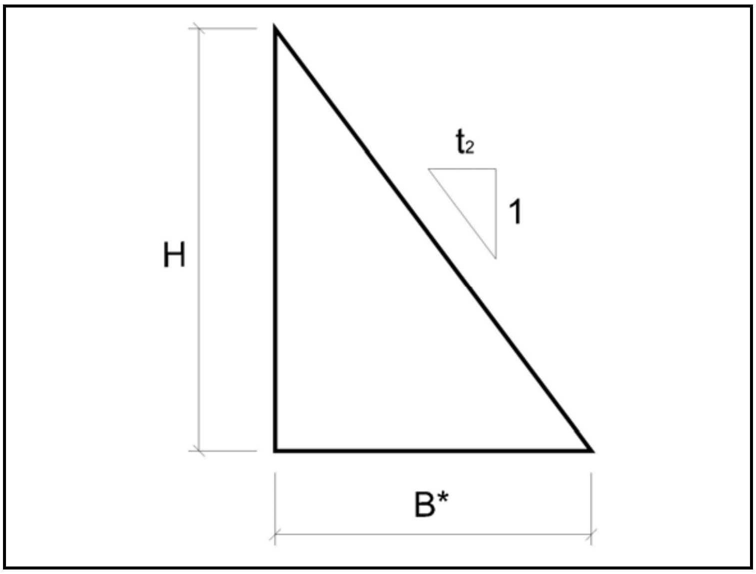

Figure 1. Dam geometry.

Table 2. Dam geometry.

Geometry	Values
Height (m)	100
Base width (m)	75
Upstream slope	Vertical
Downstream slope (H:V)	0.75

Properties of concrete and rock materials are given in Table 3.

Table 3. Concrete and rock properties.

Material properties	Concrete	Rock
Mass density (kg/m^3)	2300	2600
Compressive strength (Pa)	200×10^5	300×10^5
Tensile strength (Pa)	20×10^5	25×10^5

Properties of the dam-foundation contact are given in Table 4.

Table 4. Properties of the dam-foundation contact.

Material properties	Values
Peak Cohesion (Pa)	5×10^5
Residual Cohesion (Pa)	0
Peak Friction Angle (º)	45
Residual Friction Angle (º)	35
Tensile strength (Pa)	4×10^5

Data of water pressures are given in Table 5.

Table 5. Data of water pressures.

Data of water pressures	Values
Density of water, ρw (kg/m^3)	1000
Water level upstream, h (m)	90
Water level downstream (m)	0
Drainage system efficiency	0

Gravity acceleration is taken as g = 10 m/s^2.

3.- MATHEMATICAL MODEL OF ANALYSIS.

Sliding stability can be analysed by means of a simple two-dimensional limit equilibrium model.

Hydrostatic load, S (N/m), is the driving force and can be evaluated by (5).

$$S = \frac{1}{2}\rho_w g h^2 \qquad \text{(Eq. 5)}$$

Shear strength, R(N/m) is calculated with (6).

$$R = (N - U)tg\varphi + B \times c \qquad \text{(Eq. 6)}$$

N (N/m) is the sum of vertical forces acting on the dam-foundation contact surface.
U (N/m) is the uplift.
B (m^2/m) is the area in compression in the dam-foundation contact.
φ (º) is the friction angle in the contact.
c (Pa) is the cohesion in the contact.

4.- LEVEL 1 METHODS. GLOBAL SAFETY FACTOR

4.1.- THEORETICAL BASIS.

This is the classical approach in structural safety assessment. All the variables of a certain problem (geometry, material properties, loads,...) form a vector $(X_1,X_2,...,X_n)$ in a n-dimensional space, and if we define a strength function $r(x_1,x_2,...,x_n)$ and a loading function $s(x_1,x_2,...,x_n)$, it is possible to derive a function $g(x_1,x_2,...,x_n)$ as:

$$g(x_1,x_2,...,x_n)=\frac{r(x_1,x_2,...,x_n)}{s(x_1,x_2,...,x_n)} \qquad \text{(Eq. 7)}$$

Any point $(x_1,x_2,...,x_n)$ in the n-dimensional space is in the safety domain if:

$$g(x_1,x_2,...,x_n)>1 \qquad \text{(Eq. 8)}$$

On the other hand, it is in the failure domain if:

$$g(x_1,x_2,...,x_n)\leq 1 \qquad \text{(Eq. 9)}$$

The frontier between these two domains, or limit state region, is defined by the n-dimensional hyper surface defined by:

$$g(x_1,x_2,...,x_n)=1 \qquad \text{(Eq. 10)}$$

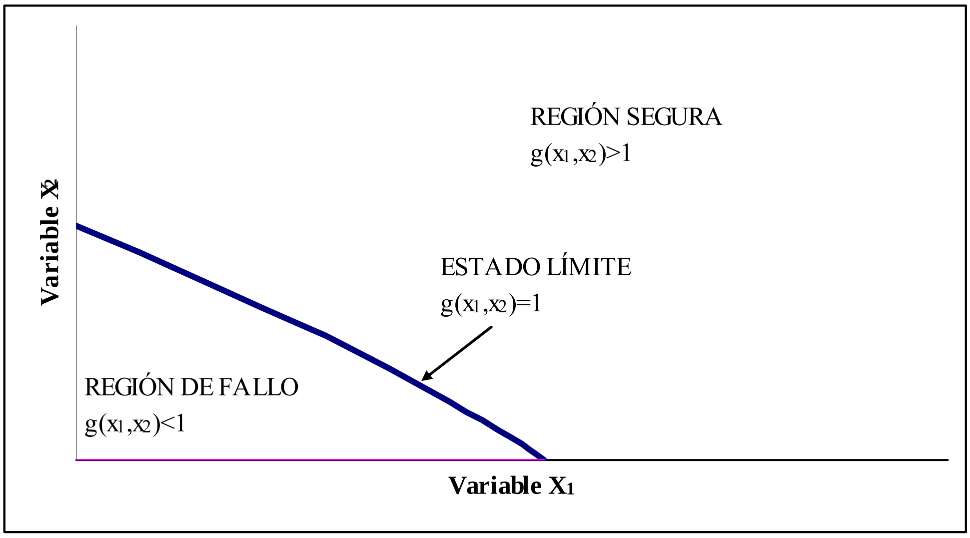

Figure 2: Safety and failure domains and limit state in a two-dimensional case.

The global safety factor, F (F>1), is defined as:

$$g(x_1,x_2,...,x_n)-F>0 \qquad \text{(Eq. 11)}$$

Or, in the most common expression:

$$\frac{r(x_1, x_2, ..., x_n)}{s(x_1, x_2, ..., x_n)} > F \qquad \text{(Eq. 12)}$$

This method is used in common practice with constant values for the variables ($X_1, X_2, ..., X_n$), so-called representative values.

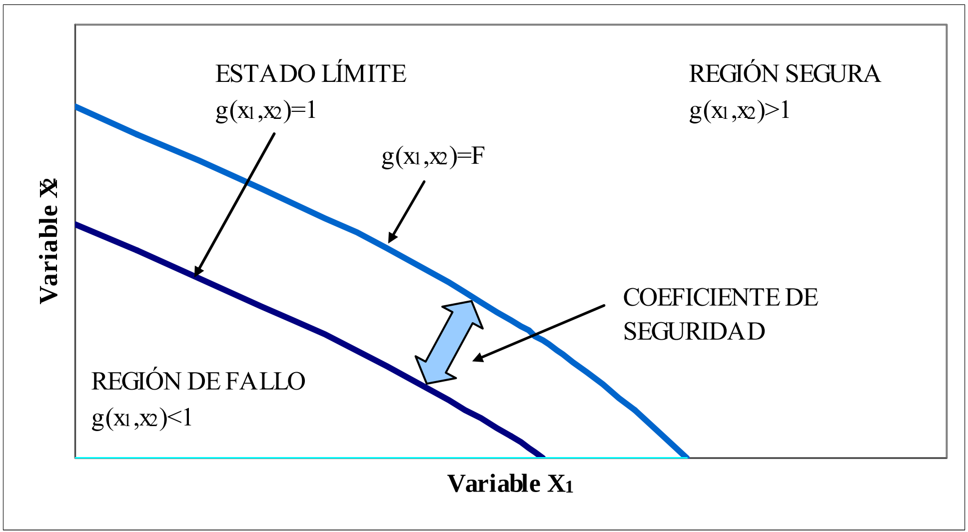

Figure 3. Safety margin expressed in terms of safety factor.

4.2.- APPLICATION TO THE CASE STUDY.

Evaluating the forces acting on the dam:

$$S = \frac{1}{2} 1000 \times 10 \times 90^2 = 4.05 \times 10^7 \text{ (N/m)} \qquad \text{(Eq. 13)}$$

$$R = ((0.5 \times 75 \times 100 \times 2300 \times 10) - ((0.5 \times 75 \times 90 \times 1000 \times 10) tg 45 + (75 \times 5 \times 10^5))) = 9.00 \times 10^7 \text{ (N/m)} \qquad \text{(Eq. 14)}$$

And the global safety factor obtained is:

$$FS = \frac{R}{S} = \frac{9.00 \times 10^7}{4.05 \times 10^7} = 2.22 \qquad \text{(Eq. 15)}$$

5.- LEVEL 1 METHODS. PARTIAL SAFETY FACTORS

5.1.- THEORETICAL BASIS.

In this methodology different safety factors are associated with different variables. This method is common practice in the reinforced concrete and steel structures analysis.

Two groups of coefficients are defined, a group γ_i (γ_i <1) associated with strength variables, R_i, and a group λ_j (λ_j > 1) associated to loadings, S_j, so (Eq. 12) can be re-written as:

$$\sum_i \gamma_i R_i > \sum_j \lambda_j S_j \quad \text{(Eq. 16)}$$

This methodology allows "weighting" of the different variables depending upon the uncertainties associated to the representative values adopted. Coefficients associated to strength variables decrease their values with respect to their representative ones and coefficients associated to loading variables increase their values with respect to their representative values.

5.2.- APPLICATION TO THE CASE STUDY.

It is possible to evaluate the sliding safety of a gravity dam with partial safety factors, as the Spanish recommendations for dam calculation state (Technical Guide nº2 Criteria for dam project).

Partial safety factors are assigned to shear strengths (friction and cohesion in the dam-foundation contact) The values for these factors are different depending upon the kind of evaluation being carried out: normal, abnormal or extreme. They also vary depending on the dam classification.

In this case assumption of an abnormal situation is reasonable, as the drainage system is supposed to be ineffective. Dam classification according to Spanish standards is A.

Friction strength, R_1, cohesion strength, R_2, and loading, S_1, can be calculated:

$$R_1 = (A \cdot \rho_c g - U) tg\phi = (3750 \times 2300 \times 10 - 3.375 \times 10^7) tg45 = 5.25 \times 10^7 \text{ N/m} \quad \text{(Eq. 17)}$$

$$R_2 = B \cdot c = 75 \times 5.00 \times 10^5 = 3.75 \times 10^7 \text{ N/m} \quad \text{(Eq. 18)}$$

$$S_1 = 0.5 \cdot \rho_w g \cdot h^2 = 0.5 \times 1000 \times 10 \times 90^2 = 4.05e7 \text{ N/m} \quad \text{(Eq. 19)}$$

and equation (16) can be written as:

$$\gamma_1 R_1 + \gamma_2 R_2 > \lambda_1 S_1 \quad \text{(Eq. 20)}$$

According to Spanish recommendations, partial safety factors for abnormal situation and A category dam are:

Friction, γ_1 = 1/1.2 = 0.833
Cohesion, γ_2 = 1/4 = 0.25

Cohesion decrease is larger than friction decrease. Recommendations do not assign any loading increase, so λ = 1. Substituting in (20) the sliding safety can be checked::

$$0.833 \times 5.25 \times 10^7 + 0.25 \times 3.75 \times 10^7 = 5.31 \times 10^7 > 1 \times 4.05 \times 10^7 \quad \text{(Eq. 21)}$$

6.- LEVEL 2 METHODS. THEORETICAL BASIS.

Level 2 methods make a linear (first order) approximation of the function $g^*(x_1,x_2,...,x_n)$. In addition, only the first two moments (second moment) of the joint probability density function distribution are considered, so these methods are called FOSM Methods (First Order Second Moment).

The typical output of these methods is the reliability index, β, which is defined as the number of standard deviations between the expected value of the function $g^*(x_1,x_2,...,x_n)$ and the limit state value defined as $g^*(x_1,x_2,...,x_n)=0$. This value gives us a relative measure of reliability (distance between the most probable value and the failure domain, in the sense that the larger the value of β, the safer the structure will be, but it does not tells us anything about the probability of failure by itself.

$$\beta = \frac{E[g*] - (g*)_{fallo}}{\sigma_{g*}} = \frac{E[g*] - 0}{\sigma_{g*}} = \frac{E[g*]}{\sigma_{g*}} \qquad \text{(Eq. 22)}$$

As $X_1,X_2,...,X_n$ are random variables, $g^*(x_1,x_2,...,x_n)$ is a random variable with a certain probability distribution, usually unknown. To get an estimate of the probability of failure, and hypothesis on the shape of this distribution is to be done. With the shape of the probability distribution and its first two moments, both the reliability index and the probability of failure can be obtained.

There are different techniques to deal with the problem:

- Taylor's Series Method
- Rosenblueth's Point Estimate Method
- Hasofer & Lind Method

7.- LEVEL 2 METHODS. FOSM – TAYLOR'S SERIES.

7.1.- THEORETICAL BASIS.

The function $g^*(x_1,x_2,...,x_n)$ must be linear to obtain the first two moments of the probability distribution of $g^*(x_1,x_2,...,x_n)$ from the first two moments of the probability distributions of the random variables $X_1,X_2,...,X_n$:

$$g*(x_1, x_2, ..., x_n) = a_0 + a_1 x_1 + a_2 x_2 + ... + a_n x_n \qquad \text{(Eq. 23)}$$

The first moment of the probability distribution of g*, assuming that the random variables are correlated can be calculated as:

$$E[g*] = g*(E[X_1], E[X_2], ..., E[X_n]) + \frac{1}{2}\sum\left(\frac{\partial^2 g*}{\partial X_i \partial X_j}\rho_{X_i X_j}\sigma_{X_i}\sigma_{X_j}\right) \qquad \text{(Eq. 24)}$$

where σ_{Xi} is the standard deviation of the random variable X_i and ρ_{XiXj} is the correlation coefficient between random variables X_i y X_j.

Being a first order approximation, second order derivatives can be ignored, so the final expression is the same for correlated and independent random variables:

$$E[g*] = g*\left(E[X_1], E[X_2], ..., E[X_n]\right) \qquad \text{(Eq. 25)}$$

So the expected value of g* is obtained evaluating the function in the n-dimensional point corresponding to the expected values of the random variables. The variance of the probability distribution of g*, assuming correlated random variables, is calculated as:

$$\mathrm{Var}[g*] = \sum_i \left(\left(\frac{\partial g*}{\partial X_i} \right)^2 \sigma_{X_i}^2 \right) + 2 \sum_{i \neq j} \left(\frac{\partial g*}{\partial X_i} \frac{\partial g*}{\partial X_j} \rho_{X_i X_j} \sigma_{X_i} \sigma_{X_j} \right) \qquad \text{(Eq. 26)}$$

where σ^2_{Xi} is the variance of the random variable X_i.

If the random variables are independent, equation (26) can be written as:

$$\mathrm{Var}[g*] = \sum_i \left(\left(\frac{\partial g*}{\partial X_i} \right)^2 \sigma_{X_i}^2 \right) \qquad \text{(Eq. 27)}$$

First order derivatives can be obtained straightforward if g* is a linear function. If it is not, first order derivatives are approximated by the first order elements of the Taylor's series expansion of g* about the expected values. The partial derivatives are calculated numerically using a very small increment (positive and negative) centred on the expected value. Following the USACE practice, a large increment of one standard deviation will be used, in order to capture some of the behaviour of the nonlinear functions.

$$\frac{\partial g^*}{\partial X_i} \approx \frac{g^*\left(E[X_i] + \sigma_{X_i}\right) - g^*\left(E[X_i] - \sigma_{X_i}\right)}{\left(X_i + \sigma_{X_i}\right) - \left(X_i - \sigma_{X_i}\right)} = \frac{g^*\left(E[X_i] + \sigma_{X_i}\right) - g^*\left(E[X_i] - \sigma_{X_i}\right)}{2\sigma_{X_i}} \qquad \text{(Eq. 28)}$$

And the square of the first order derivative can be estimated by:

$$\left(\frac{\partial g*}{\partial X_i} \right)^2 \approx \frac{1}{\sigma_{X_i}^2} \left(\frac{g*\left(E[X_i] + \sigma_{X_i}\right) - g*\left(E[X_i] - \sigma_{X_i}\right)}{2} \right)^2 \qquad \text{(Eq. 29)}$$

Substituting (29) in (27):

$$\mathrm{Var}[g*] = \sum_i \left(\left(\frac{g*\left(E[X_i] + \sigma_{X_i}\right) - g*\left(E[X_i] - \sigma_{X_i}\right)}{2} \right)^2 \right) \qquad \text{(Eq. 30)}$$

With this method a number of 2n+1 evaluations of the performance function g* is needed, being n the number of random variables considered.

7.2.- APPLICATION TO THE CASE STUDY.

In this example only two variables are considered as random variables: friction angle and cohesion along the dam-foundation contact. All the other variables are considered as constant variables with their respective constant values. Friction angle is supposed to be defined by a normal probability function, with mean of 45º and standard deviation of 6.75º. Cohesion is normally distributed with mean of 5.00×10^5 N/m^2 and standard deviation of 1.25×10^5 N/m^2. Both variables are independent.

The performance function g is defined as:*

$$g^* = \frac{r}{s} - 1 \qquad \text{(Eq. 31)}$$

Where r is the shear strength function and s is the loading function. According to the values of the case study:

$$\begin{aligned} r &= (A\cdot\rho_c g - U)tg\phi + B\cdot c = (3750\times2300\times10 - 3.375\times10^7)tg\phi + 75\times c \\ r &= 5.25\times10^7 \times tg\phi + 75\times c \end{aligned} \qquad \text{(Eq. 32)}$$

And the loading function:

$$s = 0.5\cdot\rho_w g\cdot h^2 = 0.5\times1000\times10\times90^2 = 4.05\times10^7 \text{ N/m} \qquad \text{(Eq. 33)}$$

So the performance function can be written as:

$$g^*(\varphi,c) = \frac{5.25\times10^7\times tg\phi + 75\times c}{4.05\times10^7} - 1 \qquad \text{(Eq. 34)}$$

The $tg\varphi$ introduces a non linearity in the function. First moment of g, according to (25), is:*

$$E[g^*] = \frac{5.25\times10^7\times tg45 + 75\times5.00\times10^5}{4.05\times10^7} - 1 = 1.222222 \qquad \text{(Eq. 35)}$$

And 4 more evaluations of g are needed:*

$$\begin{aligned} g^*(45+6.75, 5.00e5) &= \frac{5.25\times10^7\times tg51.75 + 75\times5.00\times10^5}{4.05e7} - 1 = 1.570270 \\ g^*(45-6.75, 5.00e5) &= \frac{5.25\times10^7\times tg38.25 + 75\times5.00\times10^5}{4.05e7} - 1 = 0.947844 \\ g^*(45, 5.00e5+1.25e5) &= \frac{5.25\times10^7\times tg45 + 75\times6.25\times10^5}{4.05e7} - 1 = 1.453704 \\ g^*(45, 5.00e5-1.25e5) &= \frac{5.25\times10^7\times tg45 + 75\times3.75\times10^5}{4.05e7} - 1 = 0.990741 \end{aligned} \qquad \text{(Eq. 36)}$$

And applying (30):

$$\mathrm{Var}[g^*] = \left(\frac{1.570270 - 0.947844}{2}\right)^2 + \left(\frac{1.453704 - 0.990741}{2}\right)^2 \qquad \text{(Eq. 37)}$$
$$\mathrm{Var}[g^*] = 0.096854 + 0.053584 = 0.150438$$

Once the first two moments are known, the reliability index can be calculated using equation (22):

$$\beta = \frac{E[g^*] - (g^*)_{\mathrm{fallo}}}{\sigma_{g^*}} = \frac{E[g^*] - 0}{\sigma_{g^*}} = \frac{E[g^*]}{\sigma_{g^*}} = \frac{1.222222}{\sqrt{0.150437}} = 3.151174 \qquad \text{(Eq. 38)}$$

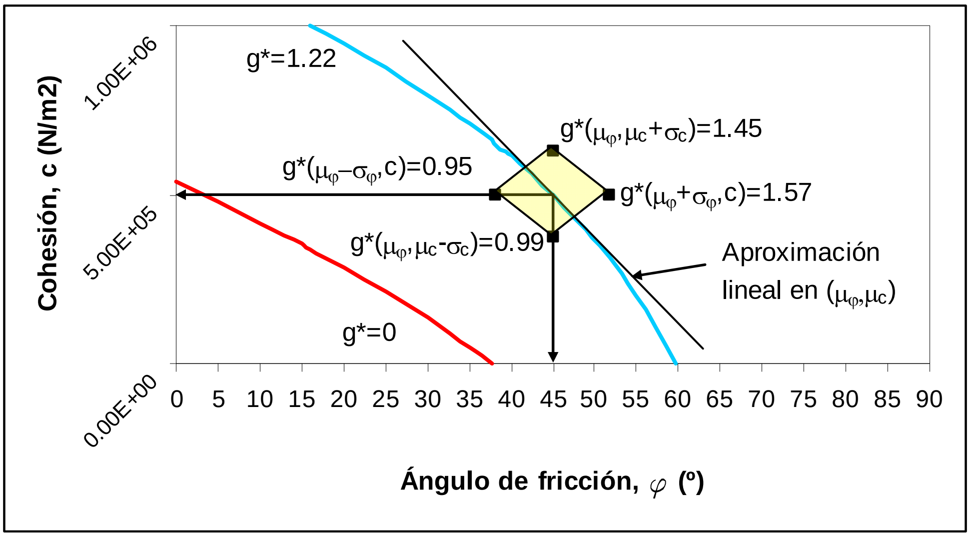

Figure 4: Taylor's series method.

Contribution of each random variable to overall variance is:

Contribution of φ: (0.096854/0.150438)→ 64.38%
Contribution of c: (0.053584/0.150438)→ 35.62%

To obtain a probability of failure, we have to make an assumption on how the performance function g is distributed. If the hypothesis is that g* is normally distributed, then:*

$$g^* \sim N(\mu_{g^*}; \sigma^2_{g^*}) \sim N(1.222222;\ 0.150438)$$

and the probability of failure $P_f[g^* \leq 0]$ can be calculated:

$$P_f[g^* \leq 0] = F_N(0) = \Phi\left(\frac{0 - \mu_{g^*}}{\sigma_{g^*}}\right) = \Phi(-\beta) = \Phi(-3.151174) = 0.000813 \qquad \text{(Eq. 39)}$$

Note that this is a CONDITIONAL probability for a certain water level upstream and certain drainage system efficiency.

8.- LEVEL 2 METHODS. POINT ESTIMATE METHOD.

8.1.- THEORETICAL BASIS.

The point estimated method determine the first two moments of the performance function g* by the discretization of the probability distributions of the random variables $X_1, X_2, ..., X_n$. This discretization is made in a few points for each random variable (two or three points), where mass probability is concentrated in such a fashion that the sum of the probabilities assigned to each point is 1 for each random variable (Rosenblueth [2] y Harr [3]). In the more general approximation, the method determines the third moment of the distributions, which allows analysis with skewed (asymmetrical) distributions. Random variables can be independent or correlated.

With this method there is no need to evaluate partial derivatives of the performance function g*. A disadvantage of the method is that the performance function has to be evaluated 2^n times, being *n* the number of random variables. If *n* is large, the method requires a considerable computational effort, above all if g* evaluation is not straightforward.

The method concentrates the mass probability of the random variable X_i in two points, x_{i+} y x_{i-}, each of them with a mass probability of P_{i+} and P_{i-}. Points are centred about the mean value, μ_{Xi}, at a distance of d_{i+} and d_{i-} times the standard deviation σ_{Xi} , respectively.

$$
\begin{aligned}
&P_{i+} + P_{i-} = 1 \\
&x_{i+} = \mu_{X_i} + d_{i+} \cdot \sigma_{X_i} \\
&x_{i-} = \mu_{X_i} + d_{i-} \cdot \sigma_{X_i}
\end{aligned}
\qquad \text{(Eq. 40)}
$$

Coefficients d_{i+} y d_{i-} are determined using the skew coefficient, γ_i, of the random variable X_i:

$$
\begin{aligned}
&d_{i+} = \frac{\gamma_i}{2} + \sqrt{1 + \left(\frac{\gamma_i}{2}\right)^2} \\
&d_{i-} = d_{i+} - \gamma_i
\end{aligned}
\qquad \text{(Eq. 41)}
$$

Probabilities are assigned to each point according to:

$$
\begin{aligned}
&P_{i+} = \frac{d_{i-}}{d_{i+} + d_{i-}} \\
&P_{i-} = 1 - P_{i+}
\end{aligned}
\qquad \text{(Eq. 42)}
$$

In figures 5 and 6 the discretization of a random variable is shown. A number of 2^n values of discrete probabilities should be obtained by combination of the point probabilities of each random variable with the other random variable's probabilities. These probabilities are $P_{(\delta 1, \delta 2, \ldots, \delta n)}$, where δ_i is the sign (+ ó -).

Values of these probabilities are calculated as:

$$
P_{(\delta 1, \delta 2, \ldots, \delta n)} = \prod_{i=1}^{n} P_{i,\delta i} + \sum_{i=1}^{n-1}\left(\sum_{j=i+1}^{n} \delta_i \delta_j a_{ij}\right) \qquad \text{(Eq. 43)}
$$

Where the coefficients a_{ij} are calculated as:

$$a_{ij} = \frac{\frac{\rho_{ij}}{2^n}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(\frac{\gamma_i}{2}\right)^2\right)}} \qquad \text{(Eq. 44)}$$

Being ρ_{ij} the correlation coefficient between random variables X_i y X_j.

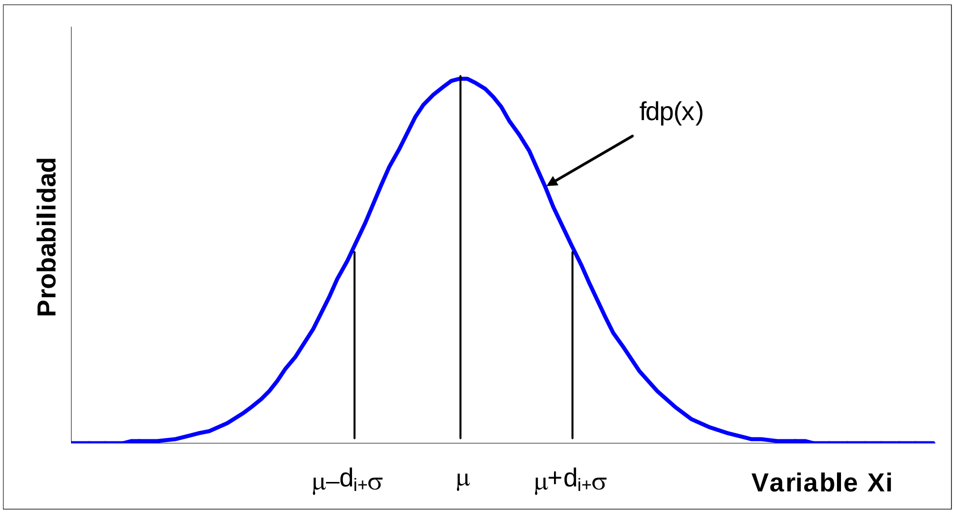

Figure 5: Probability density function of a random variable X_i.

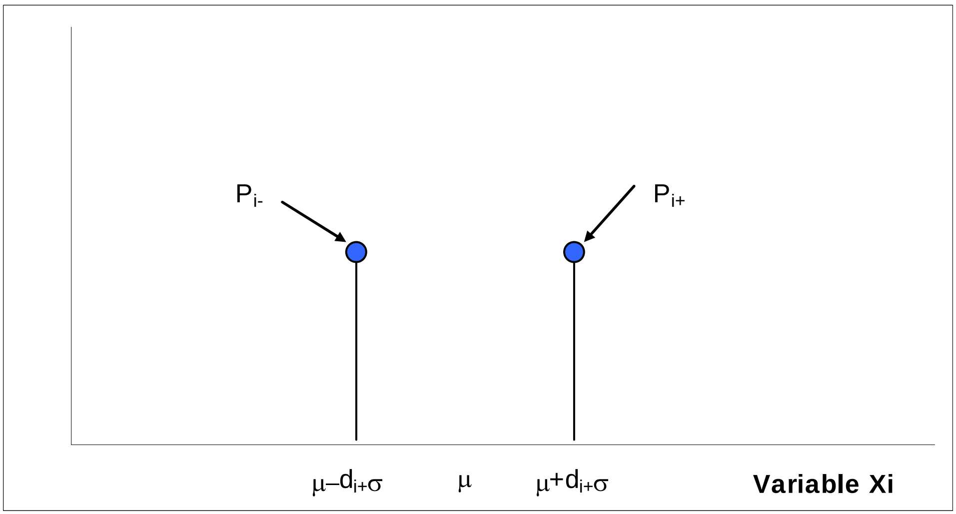

Figure 6: Point Estimate Method discretization of probability of a random variable.

The performance function g* has to be evaluated 2^n times, corresponding to the 2^n possible combinations of discrete probability points $P_{(\delta1,\delta2,\ldots,\delta n)}$, obtaining $g^*_{(\delta1,\delta2,\ldots,\delta n)}$. Once this is accomplished, the moment of *m* order of the probability distribution of g* is determined by:

$$E\left[g^{*m}\right] \approx \sum P_{(\delta1,\delta2,\ldots,\delta n)} g^{*m}_{(\delta1,\delta2,\ldots,\delta n)} \qquad \text{(Eq. 45)}$$

So for the first moment:

$$E[g*] = \sum P_{(\delta 1,\delta 2,...,\delta n)} g*_{(\delta 1,\delta 2,...,\delta n)} \quad \text{(Eq. 46)}$$

And for the second moment:

$$E[g*^2] = \sum P_{(\delta 1,\delta 2,...,\delta n)} g*^2_{(\delta 1,\delta 2,...,\delta n)} \quad \text{(Eq. 47)}$$

The variance of g* can be calculated:

$$Var[g*] = E[(g* - \mu_{g*})^2] = E[g*^2] - \mu^2_{g*} \quad \text{(Eq. 48)}$$

So it is possible to determine the mean and the variance of the probability distribution of g* but the shape of the distribution is not known. If what we want is a probability of failure, again a hypothesis of how g* is distributed is to be done.

The method loses precision with the increasing nonlinearity of g* and if moments over the second are to be obtained. It does not provide a measure of the contribution of each random variable to the overall variance, so it is not an adequate method to filter the most relevant random variables.

8.2.- APPLICATION TO THE CASE STUDY.

First step is to make the discretization of probability distributions of the random variables. Variables φ and c are normally distributed (symmetrical), so $\gamma_\varphi = \gamma_c = 0$ (null skewness). Applying (41) we can obtain $d_{\varphi+} = d_{\varphi-} = 1$ and $d_{c+} = d_{c-} = 1$, and the points where mass probabilities will concentrate are, using (40):

$$\begin{aligned} \phi_+ &= \mu_\phi + \sigma_\phi = 45 + 6.75 = 51.75 \\ \phi_- &= \mu_\phi - \sigma_\phi = 45 - 6.75 = 38.25 \\ c_+ &= \mu_c + \sigma_c = 5.00 \times 10^5 + 1.25 \times 10^5 = 6.25 \times 10^5 \\ c_- &= \mu_c - \sigma_c = 5.00 \times 10^5 - 1.25 \times 10^5 = 3.75 \times 10^5 \end{aligned} \quad \text{(Eq. 49)}$$

And mass probability values for each random variable are, using (42):

$$\begin{aligned} P_{\varphi+} &= \frac{d_{\varphi-}}{d_{\varphi+} + d_{\varphi-}} = \frac{1}{1+1} = 0.5 \\ P_{\varphi-} &= 1 - P_{\varphi+} = 1 - 0.5 = 0.5 \\ P_{c+} &= \frac{d_{c-}}{d_{c+} + d_{c-}} = \frac{1}{1+1} = 0.5 \\ P_{c-} &= 1 - P_{c+} = 1 - 0.5 = 0.5 \end{aligned} \quad \text{(Eq. 50)}$$

As friction and cohesion are supposed to be independent variables, correlation coefficient is null ($\rho_{\varphi c} = 0$), and applying (44), $a_{\varphi c} = 0$.

So the calculation of the $2^n = 2^2 = 4$ probabilities given in (43) are as follows:

$$
\begin{aligned}
P_{(\varphi+,c+)} &= P_{\varphi+} \cdot P_{c+} = 0.5 \times 0.5 = 0.25 \\
P_{(\varphi+,c-)} &= P_{\varphi+} \cdot P_{c-} = 0.5 \times 0.5 = 0.25 \\
P_{(\varphi-,c+)} &= P_{\varphi-} \cdot P_{c+} = 0.5 \times 0.5 = 0.25 \\
P_{(\varphi-,c-)} &= P_{\varphi-} \cdot P_{c-} = 0.5 \times 0.5 = 0.25
\end{aligned} \qquad \text{(Eq. 51)}
$$

The evaluation of the performance function g in the $2^n = 2^2 = 4$ points where probabilities have been calculated leads to:*

$$
\begin{aligned}
g^*(\phi_+, c_+) &= g^*(51.75, 6.25\times10^5) = \frac{5.25\times10^7 \times tg51.75 + 75\times6.25\times10^5}{4.05\times10^7} - 1 = 1.801751 \\
g^*(\phi_+, c_-) &= g^*(51.75, 3.75\times10^5) = \frac{5.25\times10^7 \times tg51.75 + 75\times3.75\times10^5}{4.05\times10^7} - 1 = 1.338788 \\
g^*(\phi_-, c_+) &= g^*(38.25, 6.25\times10^5) = \frac{5.25\times10^7 \times tg38.25 + 75\times6.25\times10^5}{4.05\times10^7} - 1 = 1.179325 \\
g^*(\phi_-, c_-) &= g^*(38.25, 3.75\times10^5) = \frac{5.25\times10^7 \times tg38.25 + 75\times3.75\times10^5}{4.05\times10^7} - 1 = 0.716362
\end{aligned} \qquad \text{(Eq. 52)}
$$

So the first moment can be determined with equation (46):

$$
\begin{aligned}
E[g^*] &= 0.25\times1.801751 + 0.25\times1.338788 + 0.25\times1.179325 + 0.25\times0.716362 \\
E[g^*] &= 1.259057
\end{aligned} \qquad \text{(Eq. 53)}
$$

And the second moment with equation (47):

$$
\begin{aligned}
E[g^{*2}] &= 0.25\times1.801751^2 + 0.25\times1.338788^2 + 0.25\times1.179325^2 + 0.25\times0.716362^2 \\
E[g^{*2}] &= 1.735661
\end{aligned} \qquad \text{(Eq. 54)}
$$

And the variance of g is calculated with equation (48):*

$$
Var[g^*] = E[g^{*2}] - \mu_{g^*}^2 = 1.735661 - 1.259057^2 = 0.150437 \qquad \text{(Eq. 55)}
$$

To obtain a probability of failure, we have to make an assumption on how the performance function g is distributed. If the hypothesis is that g* is normally distributed, then:*

$$
g^* \sim N(\mu_{g^*}; \sigma^2_{g^*}) \sim N(1.259057;\ 0.150437)
$$

and the probability of failure $P_f[g^* \le 0]$ can be calculated:

$$
P_f[g^* \le 0] = F_N(0) = \Phi\left(\frac{0-\mu_{g^*}}{\sigma_{g^*}}\right) = \Phi\left(\frac{-1.259057}{\sqrt{0.150437}}\right) = \Phi(-3.246142) = 0.000585 \qquad \text{(Eq. 56)}
$$

Note that the probability value obtained with PEM is slightly less than the value estimated with Taylor's series and that variance is the same in both cases.

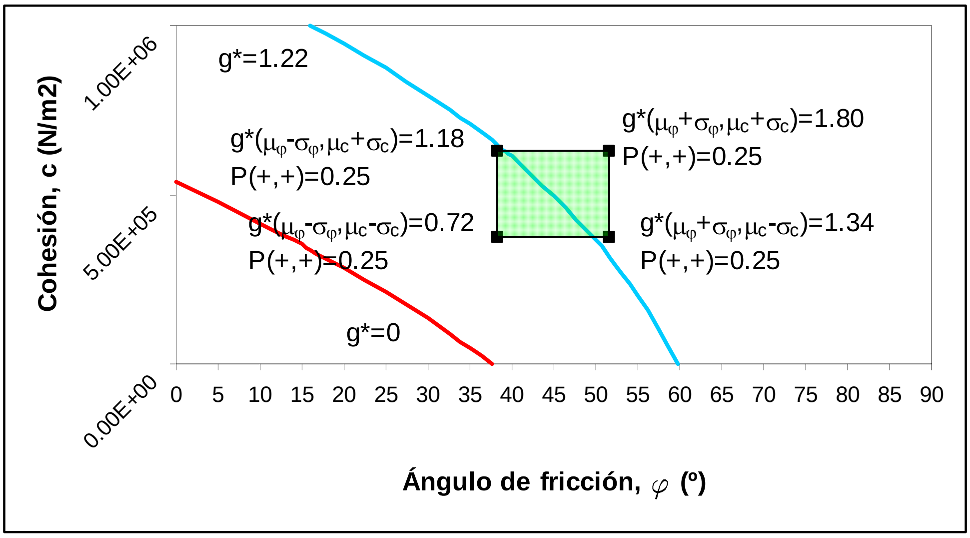

Figure 7: Point Estimate Method.

9.- LEVEL 2 METHODS. HASOFER-LIND.

9.1.- THEORETICAL BASIS.

One of the problems of the Taylor's series method and Point Estimate Method is the lack of invariance of the reliability indexes obtained, as their value depend upon the formulation of the performance function g*. To avoid this, Hasofer and Lind [4] developed an invariant definition of the reliability index.

Let **X** be the vector of the random variables $(X_1,X_2,...,X_n)$, normally distributed, $\boldsymbol{\mu_X}$ the vector of the mean values, $\boldsymbol{\sigma_X}$ the variance-covariance matrix and g^*_X the performance function, supposed to be linear. The reliability index proposed by Hasofer and Lind is defined by:

$$\beta = \underset{x}{Min}\sqrt{(x-\mu_X)^T \sigma_X^{-1}(x-\mu_X)} \qquad \text{(Eq. 57)}$$

Subject to:

$$g_X^*(x) = 0 \qquad \text{(Eq. 58)}$$

The point of the n-dimensional space that verifies the condition is the "design-point", which lies on the limit state region between the safety and the failure domains. Of all of the possible point lying on the limit state region, the design-point is the most likely. That is to say that of all possible points on the limit state region, at the design point, the joint probability density function $f_{X1,X2,..,Xn}$ of all the random variables reaches the highest value.

If random variables are independent, then the variance-covariance matrix is a diagonal matrix, where values lying on the diagonal are the variances of the random values, so the problem defined by (57) y (58) can be re-written as:

$$\beta = \underset{x_i}{Min}\sqrt{\sum_{i=1}^{n}\left(\frac{x_i-\mu_{X_i}}{\sigma_{X_i}}\right)^2} \qquad \text{(Eq. 59)}$$

Subject to:

$$g_X^*(x_1, x_2, ..., x_n) = 0 \quad \text{(Eq. 60)}$$

To apply this method is a common practice to transform normal correlated random variables (X_1,X_2,...,X_n) in standard normal uncorrelated random variables with null mean and standard deviation being unity (Z_1,Z_2,...,Z_n). To keep the metric in both spaces an orthogonal transformation should be done. The first step is to transform initial normal correlated random variables in normal uncorrelated random variables (U_1,U_2,...,U_n). This is accomplished by a transformation matrix **B**:

$$\mathbf{U} = \mathbf{BX} \quad \text{(Eq. 61)}$$

As the variance-covariance matrix is symmetric and defined positive, it can be expressed as:

$$\boldsymbol{\sigma}_X = \mathbf{LL}^T \quad \text{(Eq. 62)}$$

Where **L** is a triangular matrix which can be obtained from $\boldsymbol{\sigma}_X$. The transformation matrix is determined as:

$$\mathbf{B} = \mathbf{L}^{-1} \quad \text{(Eq. 63)}$$

It can be proved that $\boldsymbol{\sigma}_U = \mathbf{I}$ (Mínguez [1]).

Variable standardization is done by:

$$Z = U - \mu_U = B(X - \mu_X) \quad \text{(Eq. 64)}$$

In the transformed space, the formulation of the problem is as follows:

$$\beta = \underset{z}{Min} \sqrt{z^T z} \quad \text{(Eq. 65)}$$

Subject to:

$$g_Z^*(z) = 0 \quad \text{(Eq. 66)}$$

In the transformed space β is the minimum distance between the origin of coordinates and the failure domain. Vector along which the distance β is defined in the transformed space has the director cosines determined by:

$$\alpha = \frac{\dfrac{\partial g_Z^*}{\partial z}}{\sqrt{\dfrac{\partial g_Z^*}{\partial z}^T \dfrac{\partial g_Z^*}{\partial z}}} \quad \text{(Eq. 67)}$$

This director cosines represent the sensitivity of the performance function g_z* to changes in the values of the variable z_i.

To solve the minimization problem different algorithms may be used (Newton, gradient, etc.).

As in previous methods, probability is derived from the reliability index making an assumption on how the performance function is distributed. If random variables are normally distributed and the performance function is linear, then it is normally distributed too.

9.2.- APPLICATION TO THE CASE STUDY.

The calculation of the reliability index with the Hasofer-Lind method is expressed by:

$$\beta = \underset{(\phi,c)}{Min}\sqrt{\left(\frac{\phi-\mu_\phi}{\sigma_\phi}\right)^2+\left(\frac{c-\mu_c}{\sigma_c}\right)^2} = \underset{(\phi,c)}{Min}\sqrt{\left(\frac{\phi-45}{6.75}\right)^2+\left(\frac{c-5.00e5}{1.25e5}\right)^2} \quad \text{(Eq. 68)}$$

Subject to:

$$g^*(\varphi,c) = \frac{5.25e7\times \text{tg}\varphi + 75\times c}{4.05e7} - 1 = 0 \quad \text{(Eq. 69)}$$

The numerical problem can be solved with different algorithms, as Newton's or gradient method. In this particular case the tool "Solver" implemented in the commercial spreadsheet Excel© has been used to solve the problem.

Initial values are $\varphi = c = 0$. Reliability index obtained is $\beta = 3.656443$, for values at the design point of $\varphi = 28.8987º$ and $c = 1.54e5\ N/m^2$.

Assuming that g is normally distributed, sliding probability of failure is determined:*

$$P_f[g^*\le 0] = F_N(0) = \Phi(-3.656443) = 0.000128 \quad \text{(Eq. 70)}$$

This value of probability is less than the values provided by previous methods, and it is a more accurate one..

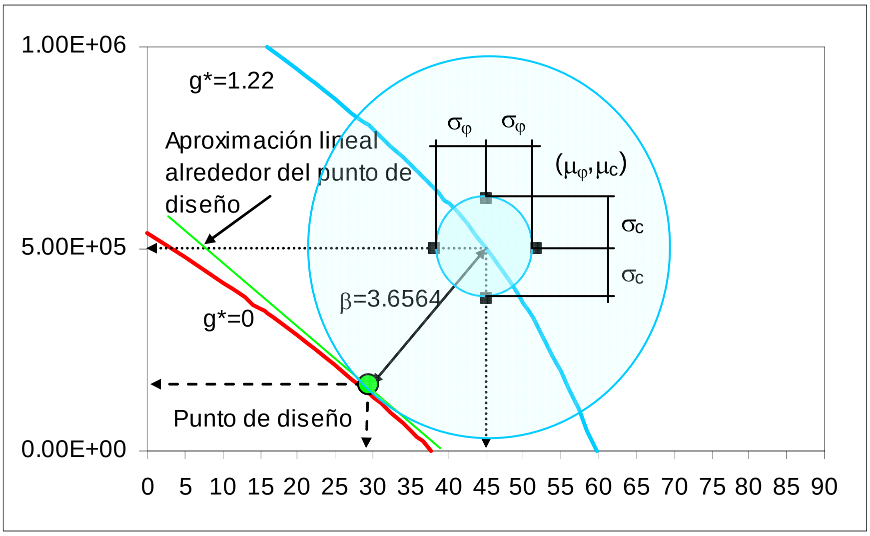

Figure 8: Hasofer-Lind method.

10.- LEVEL 3 METHODS. SIMULATION.

10.1.- THEORETICAL BASIS.

Level 3 methods provide a more accurate evaluation of the probability of failure, as they consider all the information of the probability density function and not only the first two moments. The formulation of the problem is that of equation (4).

To evaluate the integral we can resort to two groups of methods.

In the first group we can find methods based on the transformation of the random variables in a fashion similar to FOSM methods. FORM (First Order Reliability Methods) y SORM (Second Order Reliability Methods) are methods of this kind.

In the second group we can find methods that try to calculate directly the integral defined by equation (4). We have on one side the numerical methods of integration (Simpson, Gauss-Laguerre, Gauss-Hermite, etc.) and on the other side the simulation methods (Monte Carlo Methods).

With the simulation methods we generate N sets of values for the random variables according to their probability distributions and possible correlations:

$$\hat{x}_{(i)} = \left(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \right)_{(i)} \quad ; \mathbf{i = 1,\ldots, N} \qquad \text{(Eq. 71)}$$

Generation of these values is accomplished by statistical techniques as the inverse transform method, the composition method, the acceptance-rejection method, and others (Rubinstein [5]).

The performance function is evaluated for each one of these sets of values, and the number of failures, m, (when $g^* \leq 0$) is calculated. The probability of failure can be then estimated by:

$$P_{fallo} \approx \frac{m\left(g^* \left(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \right) \leq 0 \right)}{N} = \hat{P}_f \qquad \text{(Eq. 72)}$$

This method of simulation is the normal Monte Carlo method ("Hit or Miss Monte Carlo Method"). These simulation methods are deemed "exact methods" in the sense that they provide the exact value of the probability of failure when $N \to \infty$. For lower values of N, what we get is an approximation of the value of the integral (4). The estimator of the probability of failure shows a mean and a variance given by:

$$E\left[\hat{P}_f \right] = P_f$$
$$\sigma^2_{\hat{P}_f} = \frac{1}{N} P_f \left(1 - P_f \right) \qquad \text{(Eq. 73)}$$

The accuracy of the estimation is measured by inverse of the standard deviation of the estimator, which is proportional to $N^{0.5}$. So we can double the precision in the approximation of the value of the probability of failure by multiplying by four the number of experiments (USACE [6]).

Probabilities of failure in civil engineering in general and in dam engineering in particular are very low, being of an order of 1 out of 10000 and less. So to capture this order of magnitude with simulation, a large number of experiments are needed, as each experiment is a Bernouilli process, with an individual probability of failure equal to the sought probability of failure.

From the early days of the development of the method, researchers have explored techniques with the aim of increasing the efficiency of the method (obtain low variances with less experiments). Between these techniques to reduce the variance we can find the “importance sampling” (Clark [7]), the “correlated sampling” (Cochran [8]), and the “stratified sampling”, being one of its variants the “Latin Hypercube Sampling” (Iman et al. [9, 10], McKay et al [11] y Startzman el al [12]).

Latin hypercube divides the probability distribution in different intervals equally distributed along the Y axis, corresponding to the cumulative probability. During the sampling process an identical number of experiments are generated on each of the intervals, so all the probability distribution space is swept, even those areas of very low probability that would not have been sampled unless a very large number of experiments had been done.

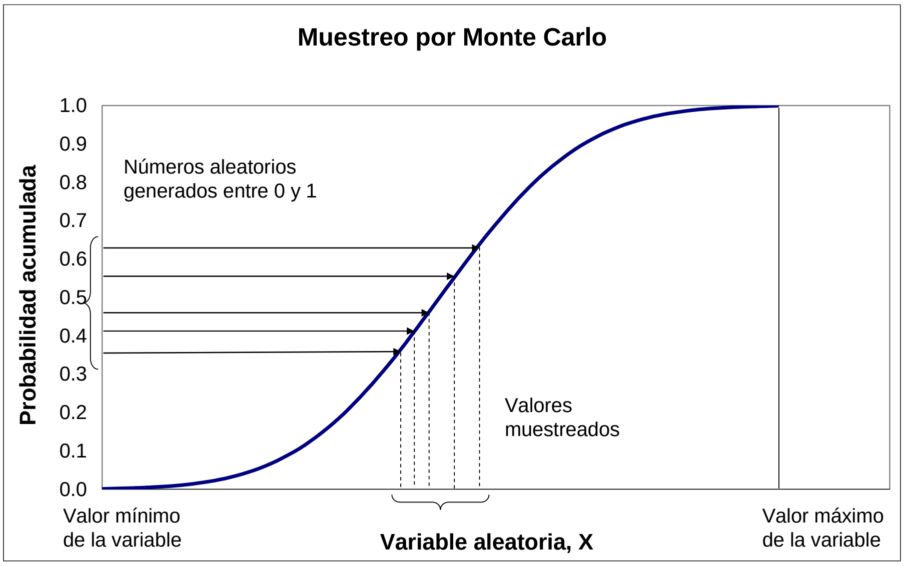

Figure 9: Monte Carlo sampling.

So it is very useful to estimate the order of magnitude of the probability of failure previously to the programming of a Monte Carlo to optimize the simulation.

It is a common practice to use Monte Carlo techniques to make inferences of the probability distribution of the performance function and of the probability distribution of the safety factor, which are closely related.

The N evaluations of the performance function form a sample of a random variable, so it is possible to make estimations on important parameters (mean, standard deviation, skewness, etc.) that help to understand how the performance function is distributed in terms of probability.

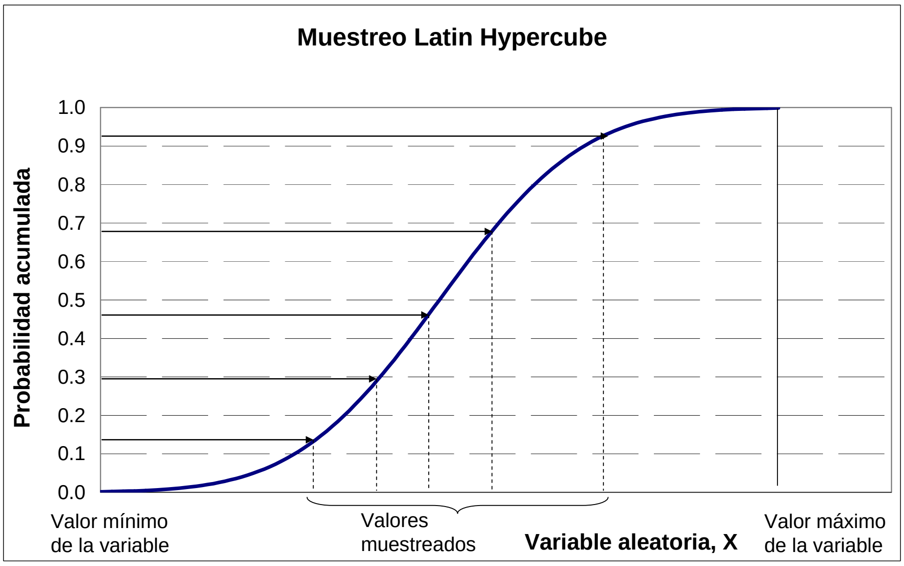

Figure 10: Latin Hypercube sampling.

Once the probability distribution function F_{g*}, of the performance function is derived, the probability of failure can be determined straightforward by:

$$P_f = P[g^* \leq 0] = F_{g^*}(0) \quad \text{(Eq. 74)}$$

An apparent advantage of this procedure is that, once F_{g*} is known, which can be done with a relative low number of experiments, N, the whole probability domain is fully determined, so probability estimations can be done at any level, even in the tails of the distributions. The problem that immediately arises is that the estimation made on the probability distribution function can be (and it usually is) inaccurate in the tails of the distribution, which are the key areas to estimate the probability of failure

10.2.- APPLICATION TO THE CASE STUDY

The failure domain g=0 is defined by equation (75):*

$$g^*(\phi, c) = a_0 + a_1 \times tg\phi + a2 \times c = -1 + 1.30 tg\phi + 1.85 \times 10^6 \times c = 0 \quad \text{(Eq. 75)}$$

We consider two random variables: friction angle and cohesion, being both normally distributed.

By Monte Carlo techniques different sets of experiments are generated. The number of experiments differs for each set: N =100, 1000, 10000, 100000 y 1000000. The sampling is done using two techniques: Monte Carlo sampling and Latin Hypercube sampling.

Each pair of sampled values will be used to evaluate the performance function, g, and so determination of the number of "failures", m, where $g^* \leq 0$ will be calculated. Probability of failure, Pf, is estimated using equation (72). The variance of the probability obtained is calculated using equation (73).*

Calculations had been carried out with the commercial tool @RISK® implemented in an Excel® spreadsheet. Results are given in Tables 6 and 7.

Table 6. Estimation of the probability of failure with Monte Carlo sampling

Simulations with Monte Carlo sampling					*Direct Integration Method*
Number of experiments, N	*Number of misses, m*	*Probability of failure, Pf = m/N*	*Variance*	*Standard deviation*	*Exact probability of failure, Pf*
1000	*0*	*0*	*0*	*0*	1.11×10^{-4}
10000	*1*	1.00×10^{-4}	1.00×10^{-8}	1.00×10^{-4}	1.11×10^{-4}
100000	*18*	1.80×10^{-4}	1.80×10^{-9}	4.24×10^{-5}	1.11×10^{-4}
1000000	*135*	1.35×10^{-4}	1.35×10^{-10}	1.16×10^{-5}	1.11×10^{-4}

Table 7. Estimation of the probability of failure with Latin Hypercube sampling

Simulations with Latin Hypercube					*Direct Integration Method*
Number of experiments, N	*Number of misses, m*	*Probability of failure, Pf = m/N*	*Variance*	*Standard deviation*	*Exact probability of failure, Pf*
1000	*0*	*0*	*0*	*0*	1.11×10^{-4}
10000	*2*	2.00×10^{-4}	2.00×10^{-8}	1.41×10^{4}	1.11×10^{-4}
100000	*10*	1.00×10^{-4}	1.00×10^{-9}	3.16×10^{-5}	1.11×10^{-4}
1000000	*116*	1.16×10^{-4}	1.16×10^{-10}	1.08×10^{-5}	1.11×10^{-4}

It should be noted that the method provides accurate results for a number of experiments of the same order of magnitude or larger that the probability of failure Latin Hypercube shows slightly better results for the same number of experiments. In Figure 11 is shown graphically the calculation of the probability of failure with Monte Carlo sampling for N=10000.

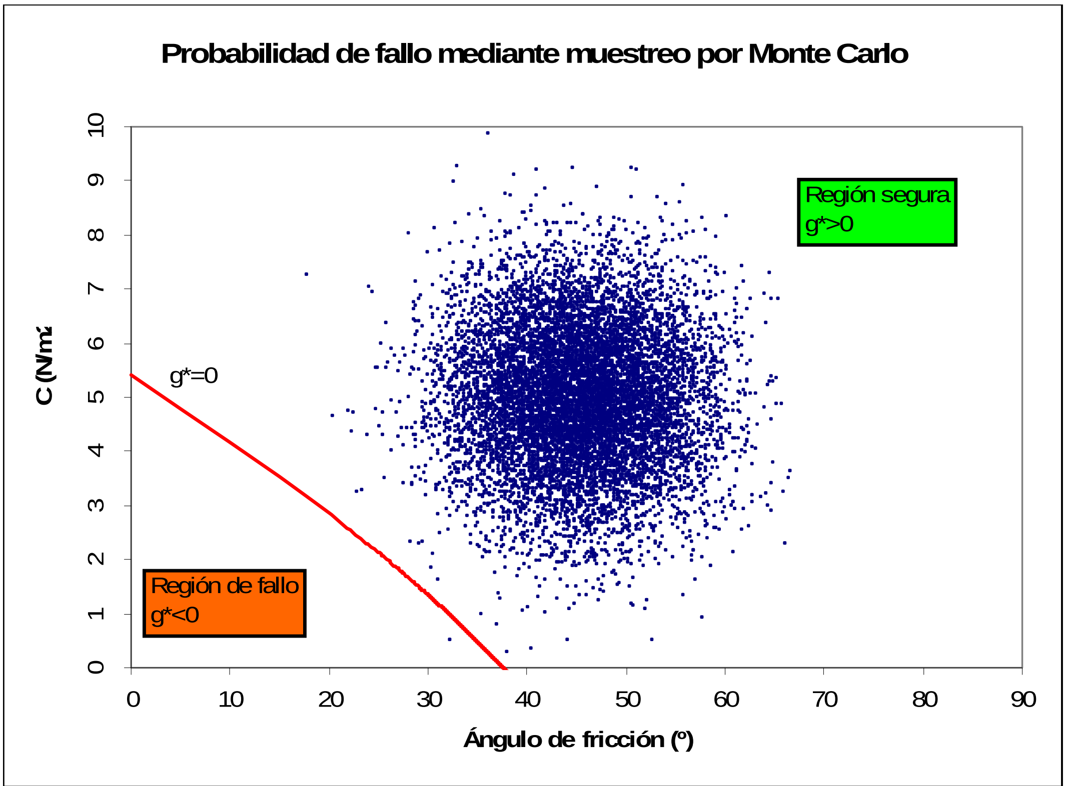

Figure 11: Probability of failure with Monte Carlo sampling.

We can try to adjust a probability distribution to the N values of g sampled. Table 8 shows the estimators for the mean, variance, standard deviation and skewness, for the sampled values using Monte Carlo sampling.*

Table 8. Estimators for parameters of g*. Monte Carlo sampling

Estimators for parameters of g*. Monte Carlo sampling				
Number of experiments, N	Mean	Variance	Standard deviation	Skewness
1000	1.278071	0.170130	0.412469	0.500776
10000	1.268753	0.170377	0.412768	0.590840
100000	1.260465	0.163442	0.404279	0.482087
1000000	1.260573	0.162467	0.403072	0.474239

Results for values sampled using Latin Hypercube techniques are given in Table 9.

Table 9. Estimators for parameters of g*. Latin Hypercube sampling

Estimators for parameters of g*. Latin Hypercube sampling				
Number of experiments, N	Mean	Variance	Standard deviation	Skewness
1000	1.260921	0.160528	0.400660	0.372136
10000	1.261030	0.163296	0.404099	0.465118
100000	1.261040	0.163101	0.403858	0.479242
1000000	1.261041	0.162702	0.403364	0.483939

It should be noted that faster convergence is obtained with Latin Hypercube.

Adjustment for two probability distribution functions has been done for the case of N=10000 experiments with both sampling methods (Monte Carlo and Latin Hypercube). This adjustment has been carried out with @RISK®. Chi-square good-of-fitness test has been carried out as well.

Two distributions have been tested: normal and lognormal. Lognormal distribution has been considered as results show certain skewness while normal distribution is symmetric.

The performance function g, defined by equation (75) can adopt both positive and negative values.*

Normal distribution is defined in the whole domain ($-\infty < g^* < +\infty$) while Lognormal distribution is defined in the positive interval ($0 \le g^* < +\infty$). This is the reason why in the process of adjustment it is necessary to consider an offset, s, so the adjustment is done for a transformed function G, defined as:*

$$G^* = g^* + s \quad ; 0 \le G^* < \infty \qquad \text{(Eq. 76)}$$

Chi-square goodness-of-fit test is based in the estimation of::

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - E_i)}{E_i} \qquad \text{(Eq. 77)}$$

where:

k = number of intervals in which is divided the domain of g (74 in this case)*
n_i = number of sampled values lying in the i interval.
E_i = expected value of the number of values corresponding to the i interval.

The better the fit between a certain probability distribution and the sampled values, the less is the value of χ^2.

One the adjustment is made, the probability of failure can be estimated with equation (74).

Results for adjustments and probabilities of failure with Monte Carlo sampling are given in Table 10. In Figures 12 and 13 the adjustment is shown graphically. The Lognormal distribution shows a better fit to sampled values.

Table 10. Fit of probability distributions to g*. Monte Carlo sampling

Values of g* evaluated with Monte Carlo sampling					
Number of experiments, N =10000					Probability of failure P ($g^* \le 0$)
Distribution	Mean	Variance	Offset	Test χ^2	
Normal	1.268753	0.170377	0	268.5	1.06×10^{-3}
Lognormal	2.728899	0.168983	-1.460194	84.5	2.07×10^{-5}

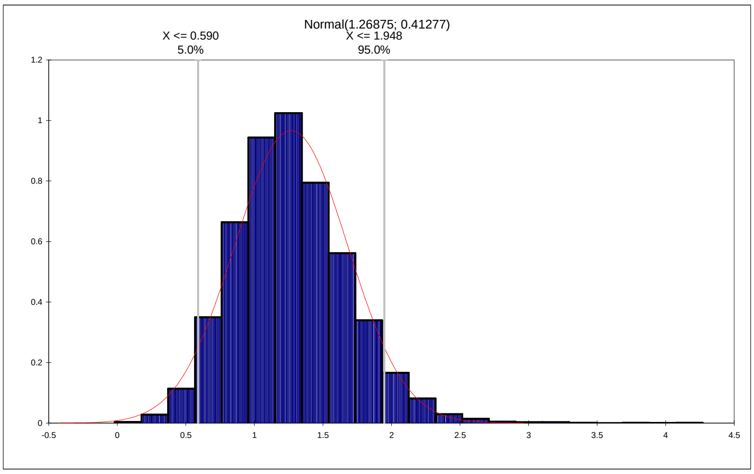

Figure 12: Fit of a normal distribution to the performance function, g*.

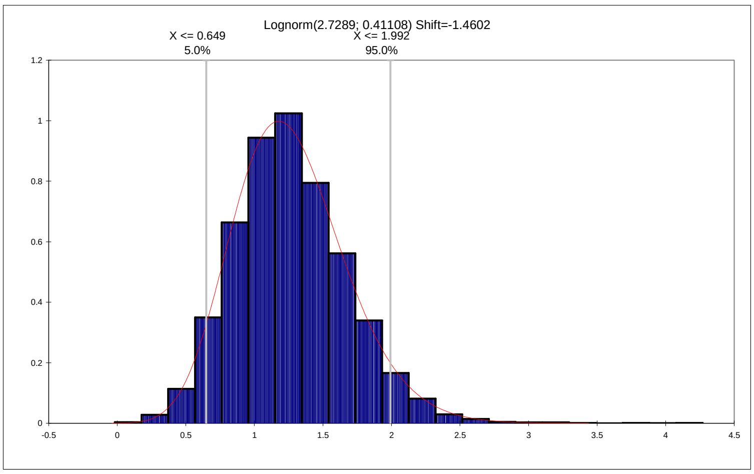

Figure 13: Fit of a lognormal distribution to the performance function, g*.

If comparison is made between the probability of failure provided by simulation for N=10000 experiments (1.41×10^{-4}) and the probability of failure estimated adjusting a normal distribution to the sampled values of g (1.06×10^{-3}) it is shown that there is an overestimation of the probability of failure. On the other hand, the probability of failure obtained adjusting a Lognormal distribution (2.07×10^{-5}) is underestimated. This illustrates the strong difficulties that come from fitting distributions to data, if the sought information is in the tails of the distributions.*

Results for adjustments and probabilities of failure with Latin Hypercube sampling are given in Table 11. In Figures 14 and 15 the adjustment is shown graphically. The Lognormal distribution shows again a better fit to sampled values.

Table 11. Fit of probability distributions to g*. Latin Hypercube sampling.

Values of g evaluated with Latin Hypercube sampling*					
Number of experiments, N =10000					*Probability of failure P ($g^*\leq 0$)*
Distribution	*Mean*	*Variance*	*Offset*	*Test χ^2*	
Normal	*1.261030*	*0.163296*	*0*	*264.8*	*9.02×10^{-4}*
Lognormal	*3.047930*	*0.162627*	*-1.786922*	*87.1*	*3.34×10^{-5}*

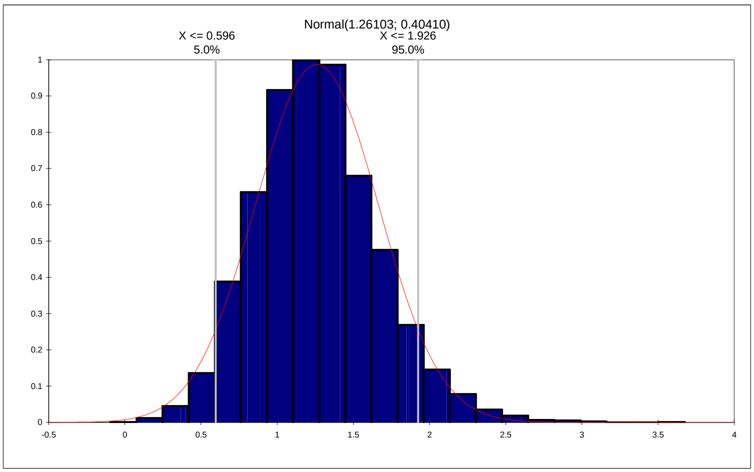

Figure 14: Fit of a normal distribution to the performance function, g*.

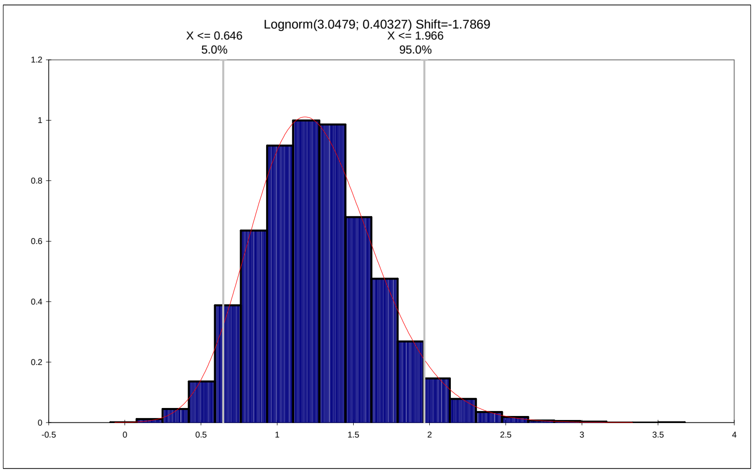

Figure 15: Fit of a lognormal distribution to the performance function, g*.

If comparison is made between the probability of failure provided by simulation for N=10000 experiments (1.41×10^{-4}) and the probability of failure estimated adjusting a normal distribution to the sampled values of g (9.02×10^{-4}) it is shown that there is an overestimation of the probability of failure. On the other hand, the probability of failure obtained adjusting a Lognormal distribution (3.34×10^{-5}) is underestimated*

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